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Dissipativity of Runge–Kutta methods for neutral delay differential equations with piecewise constant delay[☆]

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Abstract

The numerical approximation of neutral dissipative initial value problems with piecewise constant delay by fixed time stepping Runge–Kutta methods is considered and two sufficient conditions for the dissipativity of the system are given. The concept of (weak) $E(\lambda)$-dissipativity is introduced. It is shown that under some assumptions a *DJ*-irreducible, algebraically stable Runge–Kutta method is (weakly) $E(\lambda)$-dissipative.

Keywords: Neutral delay differential equations; Piecewise constant delay; Dynamical systems; Runge–Kutta methods; Dissipativity

1. Neutral delay differential equations with piecewise constant delay

The problems of interest are evolutionary problems of the type

$$y'(t) = f(t, y(t), y([t]), y'([t])), \quad t \geq 0, \tag{1.1a}$$

subject to

$$y(0) = y_0, \tag{1.1b}$$

where $[x]$ is the greatest integer less than or equal to x. Conditions will be imposed later upon f. This is a particular example of delay differential equations of the type

$$\begin{cases} y'(t) = f(t, y(t), y(\eta(t)), y'(\eta(t))), & t \geq 0, \\ y(t) = \phi(t), & t \leq 0, \end{cases} \tag{1.2}$$

which provide a mathematical instrument to applied science [1–3].

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In this work, we concentrate on the dissipativity of problem (1.1). Here the term dissipativity is used to designate that the system possesses a bounded absorbing set [4–6]. For its precise definition, we refer the reader to Section 2. To study the dissipativity of the system (1.1), we first discuss the definition of the solution because there are many ways of interpreting the problem under consideration, which may lead to a different solution, no solution or a non-unique solution. In [7], Wiener gave the definition of the solution of the linear problem

$$\begin{cases} y'(t) = Ay(t) + A_0 y([t]) + A_1 y'([t]), & t \geq 0, \\ y(0) = y_0, \end{cases} \tag{1.3}$$

and represented explicitly the solution of (1.3). Following [7], we give the following definition.

Definition 1.1. A solution of (1.1) on $[0, +\infty)$ is a continuous function $y(t)$ that satisfies the conditions:

(i) The derivative $y'(t)$ exists at each point $t \in [0, +\infty)$, with the possible exception of the points $[t] \in [0, +\infty)$ where one-sided derivatives exist.

(ii) (1.1) is satisfied on each interval $[m, m+1) \subset [0, +\infty)$ with integral end-points. Throughout this work, m denotes a non-negative integer.

Then, a unique solution can be obtained on successive intervals $[m, m + 1)$ by a "method of steps" applied to (1.1a).

Note that as a special case of (1.1a) we have the delay differential equations (DDEs) with piecewise constant delay

$$y'(t) = f(t, y(t), y([t])), \quad t \geq 0, \tag{1.4}$$

which have been widely studied by applied mathematicians and numerical researchers [8–11].

2. The dissipativity of the underlying systems

Let $\mathbf{H}$ be a real or complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$, $\mathbf{X}$ a dense continuously imbedded subspace of $\mathbf{H}$, and let $B(0, r) \equiv \{x \in \mathbf{H} : \|x\| < r\}$ for any $r > 0$, $f : [0, \infty) \times \mathbf{X} \times \mathbf{X} \times \mathbf{X} \to \mathbf{H}$ be a locally Lipschitz continuous function and satisfy one of the following two groups of conditions:

(i) Conditions 1

$$\mathcal{R}e\langle f(t, y, u, v), y\rangle \leq \alpha\|y\|^2 + \beta\|f(t, 0, u, v)\|^2,$$
$$\|f(t, 0, u, f([t], u, v, w))\|^2 \leq \gamma + \mu\|u\|^2 + \sigma\|f([t], 0, v, w)\|^2,$$

for all $t \geq 0$, $y, u, v, w \in \mathbf{X}$, where $\alpha, \beta \geq 0$, $\gamma \geq 0$, $\mu \geq 0$ and $\sigma \geq 0$ are constants.

(ii) Conditions 2

$$\mathcal{R}e\langle f(t, y, u, v), y\rangle \leq \alpha\|y\|^2 + \beta\|f(t, 0, u, v)\|^2,$$
$$\|f(t, y, u, v)\|^2 \leq \gamma_1 + L_y\|y\|^2 + \sigma\|f(t, 0, u, v)\|^2$$
$$\|f(t, 0, u, v)\|^2 \leq \gamma_2 + L_u\|u\|^2 + L_v\|v\|^2,$$

for all $t \geq 0$, $y, u, v \in \mathbf{X}$, where $\alpha, \beta \geq 0, \sigma \geq 0, \gamma_1 \geq 0, L_y \geq 0, \gamma_2 \geq 0, L_u \geq 0$ and $L_v \geq 0$ are constants.

For any $\varphi \in \mathbf{H}$ and any $t \geq 0$, we shall always assume that the equation $x = f([t], \varphi, \varphi, x)$ has a unique true solution x.

Definition 2.1 (*See Humphries and Stuart [4] or Hill [5]*). The evolutionary equation (1.1) is dissipative in $\mathbf{H}$ if there is a bounded set $B \subset \mathbf{H}$ such that for all bounded sets $D \subset \mathbf{H}$ there is a time $t_0(D)$, such that for all initial data $y_0 \in D$ the corresponding solution $y(t)$ is contained in B for all $t \geq t_0$. B is called an absorbing set in $\mathbf{H}$.

Theorem 2.1. *Suppose $y(t)$ is a solution of (1.1) where f satisfies Conditions 1 and*

$$\alpha(1 - \sigma) + \beta\mu < 0, \quad \sigma < 1. \tag{2.1}$$

Then for any $\epsilon > 0$ there exists $t^* = t^*(\|y_0\|, \epsilon)$ such that for all $t > t^*$,

$$\|y(t)\|^2 < \frac{\beta\gamma}{-((1-\sigma)\alpha + \beta\mu)} + \frac{\beta\gamma}{-(1-\sigma)\alpha}(1 - e^{2\alpha}) + \epsilon = R_1 + \epsilon. \tag{2.2}$$

Hence the system is dissipative; the open ball $B = B(0, \sqrt{R_1 + \epsilon})$ is an absorbing set for any $\epsilon > 0$.

Proof. To prove the theorem, let us define $Y(t) = \|y(t)\|^2$. Then we have

$$\begin{aligned} Y'(t) &= 2\mathcal{R}e\langle y(t), f(t, y(t), y([t]), y'([t]))\rangle \\ &\le 2\alpha Y(t) + 2\beta\|f(t, 0, y([t]), y'([t]))\|^2 \\ &\le 2\alpha Y(t) + 2\beta(\gamma + \mu\|y([t])\|^2 + \sigma\|f([t], 0, y([t]), y'([t]))\|^2). \end{aligned} \tag{2.3}$$

On the other hand, we have

$$\|f([t], 0, y([t]), y'([t]))\|^2 \le \gamma + \mu\|y([t])\|^2 + \sigma\|f([t], 0, y([t]), y'([t]))\|^2. \tag{2.4}$$

Comparing (2.3) with (2.4), we obtain

$$Y'(t) \le 2\alpha Y(t) + 2\beta\left(\frac{\gamma}{1-\sigma} + \frac{\mu}{1-\sigma}\|y([t])\|^2\right). \tag{2.5}$$

Multiply both sides by $e^{-2\alpha t}$ to obtain

$$e^{-2\alpha t}Y'(t) - e^{-2\alpha t}2\alpha Y(t) \le e^{-2\alpha t}2\beta\left(\frac{\gamma}{1-\sigma} + \frac{\mu}{1-\sigma}\|y([t])\|^2\right).$$

Then it follows that

$$\int_{t_1}^{t_2}(e^{-2\alpha x}Y(x))'dx \le \int_{t_1}^{t_2} e^{-2\alpha x}2\beta\left(\frac{\gamma}{1-\sigma} + \frac{\mu}{1-\sigma}\|y([x])\|^2\right)dx, \quad \forall\, 0 \le t_1 \le t_2 < +\infty,$$

which implies

$$Y(t_2) \le e^{2\alpha(t_2-t_1)}Y(t_1) + \frac{\beta\gamma}{-\alpha(1-\sigma)}(1 - e^{2\alpha(t_2-t_1)}) + \frac{\beta\mu}{1-\sigma}\int_{t_1}^{t_2} e^{2\alpha(t_2-x)}2Y([x])dx. \tag{2.6}$$

When $t \in [m, m+1]$, taking $t_1 = m$ and $t_2 = t$ in inequality (2.6), we obtain

$$Y(t) \le \left(\frac{\beta\mu}{-\alpha(1-\sigma)} + \frac{\alpha(1-\sigma)+\beta\mu}{\alpha(1-\sigma)}e^{2\alpha(t-m)}\right)Y(m) + \frac{\beta\gamma}{-\alpha(1-\sigma)}(1 - e^{2\alpha(t-m)}).$$

From (2.1), we further have

$$Y(t) \le Y(m) + r, \quad t \in [m, m+1), \tag{2.7}$$

and

$$Y(m+1) \le \theta Y(m) + r, \tag{2.8}$$

where

$$\theta = \frac{\beta\mu}{-\alpha(1-\sigma)} + \frac{\alpha(1-\sigma)+\beta\mu}{\alpha(1-\sigma)}e^{2\alpha}, \quad r = \frac{\beta\gamma}{-\alpha(1-\sigma)}(1 - e^{2\alpha}).$$

By simple induction, we have

$$Y(t) \le Y(m) + r \le \theta Y(m-1) + 2r \le \theta^m Y(0) + r\sum_{i=0}^{m-1}\theta^i + r. \tag{2.9}$$

Noting $0 < \theta < 1$, from (2.9) we easily obtain (2.2). This completes the proof. □

As in the proof of Theorem 2.1, the following theorem can be easily obtained.

Theorem 2.2. *Suppose $y(t)$ is a solution of (1.1) where f satisfies Conditions 2 and*

$$\alpha + \beta(L_u + L_v L_y) - L_v\sigma\alpha < 0, \quad L_v\sigma < 1. \tag{2.10}$$

Then for any $\epsilon > 0$ there exists $t^* = t^*(\|y_0\|, \epsilon)$ such that for all $t > t^*$,

$$\|y(t)\|^2 < \frac{\beta(\gamma_2 + L_v\gamma_1)}{-(\alpha + \beta(L_u + L_v L_y) - L_v\sigma\alpha)} + \frac{\beta(\gamma_2 + L_v\gamma_1)}{-(1 - L_v\sigma)\alpha}(1 - e^{2\alpha}) + \epsilon = R_2 + \epsilon. \tag{2.11}$$

Hence the system is dissipative; the open ball $B = B(0, \sqrt{R_2 + \epsilon})$ is an absorbing set for any $\epsilon > 0$.

Remark 2.1. For DDEs (1.4), the two sufficient conditions are identical and we can obtain the same result as appears in [11].

Example 2.1. Consider the problem

$$\begin{cases} y'(t) = Ay(t) + A_0 y([t]) + A_1 y'([t]) + A_2, & t \geq 0, \\ y(0) = y_0, \end{cases} \tag{2.12}$$

where A, A_0, A_1 and A_2 are complex numbers and $y(t)$ is a complex-valued scalar function. On the basis of Theorem 2.1, we can assert that the system is dissipative if there exists $\theta > 1$ such that $1 - \theta|A_1| > 0$ and $(1 - \theta|A_1|)\mathcal{R}eA + |A_0 + AA_1| < 0$. As a matter of fact, for any $0 < \delta < -((1 - \theta|A_1|)\mathcal{R}eA + |A_0 + AA_1|)$, we can choose

$$\alpha = \mathcal{R}eA + \frac{|A_0 + AA_1| + \delta}{2(1 - \theta|A_1|)}, \quad \beta = \frac{1 - \theta|A_1|}{2(|A_0 + AA_1| + \delta)}, \quad \mu = \frac{(|A_0 + AA_1| + \delta)^2}{1 - \theta|A_1|},$$
$$\sigma = \theta|A_1|, \quad \gamma = \left[\frac{1}{\kappa} + \varrho|A_1| + 1\right] A_2^2,$$

where

$$\varrho > \frac{1}{\varpi}, \quad \varpi = \min\left\{\theta - \frac{1}{\theta}, \theta - |A_1|\right\}, \quad \kappa = \frac{\theta^2\varrho|A_1| - \varrho|A_1| - \theta|A_1|}{(1 - \theta|A_1|)(\theta\varrho - 1 - \varrho|A_1|)} + \frac{2\delta}{|A_0 + AA_1|},$$

such that Conditions 1 and (2.1) hold with the standard inner product on **C**.

Example 2.2. As a specific example, consider the nonlinear problem

$$y'(t) = -ay(t) + \frac{cy'([t])}{1 + (y'([t]))^n}, \tag{2.13}$$

where $y(t)$ is a real-valued scalar function, $a > 0$ and c are real parameters and n is an even positive integer. Then, on the basis of Theorem 2.2, we can assert that the system is dissipative. As a matter of fact, for any $0 < \epsilon < a$, we can choose

$$\alpha = -a + \epsilon, \quad \beta = \frac{1}{4\epsilon}, \quad L_y = (1 + \varpi)a^2, \quad \sigma = \frac{1 + \varpi}{\varpi}, \quad \gamma_1 = 0, \quad L_u = L_v = 0, \quad \gamma_2 = c^2,$$

where $\varpi > 0$, such that Conditions 2 and (2.10) are satisfied with the usual Euclidean inner product on **R**.

3. The dissipativity of Runge–Kutta methods

In this section, we will discuss the dynamics of the solution sequence of Runge–Kutta methods applied to (1.1). Let (A, b^{T}, c) denote a given Runge–Kutta method with $s \times s$ matrix $A = (a_{ij})$ and vectors $b = [b_1, \ldots, b_s]^{\mathrm{T}}$, $c = [c_1, \ldots, c_s]^{\mathrm{T}}$. In this work we will always assume that $c_i < c_{i+1}$, $c_i \in [0, 1]$, $\forall i$, and that the method is consistent, which implies that $\sum_{i=1}^{s} b_i = 1$. For any given positive integer k, applying a Runge–Kutta method (A, b^{T}, c) with step size $h = 1/k$ to (1.1), we have

$$\begin{aligned} Y_i^{(n+1)} &= y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j^{(n+1)}, y_l, \bar{y}_l), \quad i = 1, 2, \ldots, s, \\ y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i^{(n+1)}, y_l, \bar{y}_l), \\ \bar{y}_l &= f(t_l, y_l, y_l, \bar{y}_l), \end{aligned} \tag{3.1}$$

where $t_n = nh$ $(i = 0, 1, 2, \ldots)$ are net points, y_n and $\bar{y}_n$ are approximations to the true solution $y(t_n)$ and the derivative $y'(t_n)$, and $l = k[\frac{n}{k}]$.

Definition 3.1. A method (A, b^{T}, c) is said to be $E(\lambda)$-dissipative if, whenever the method with a step size $h = 1/k$ is applied to a dynamical system of the form (1.1) subject to Conditions 1 and

$$\alpha + \frac{\beta\mu}{\lambda(1-\sigma)} < 0, \quad \sigma < 1, \tag{3.2}$$

there exists a bounded set $\mathcal{B} \subset \mathbf{H}$ such that given any solution sequence of the numerical approximation of (1.1), there is an n_0 such that $y_n \in \mathcal{B}$ for all $n \geq n_0$. The weak $E(\lambda)$- dissipativity is defined by replacing Conditions 1 and (3.2) with Conditions 2 and

$$\alpha + \frac{\beta(L_u + L_v L_y)}{\lambda(1 - L_v\sigma)} < 0, \quad L_v\sigma < 1. \tag{3.3}$$

In particular, a (weakly) $E(1)$-dissipative method is called (weakly) E-dissipative for short.

Below, we collect several definitions and results that are required in what follows.

Definition 3.2 (*See Burrage and Butcher [12] or Hairer and Wanner [13]*). A Runge–Kutta method (A, b^{T}, c) satisfying:

(i) $b_i \geq 0$ for $i = 1, 2, \ldots, s$;
(ii) $M = (m_{ij}) = (b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^{s}$ is non-negative definite,

is called algebraically stable.

Definition 3.3 (*See Hairer and Wanner [13]*). A method (A, b^{T}, c) is called *DJ*-reducible if for some nonempty index set $T \subset \{1, \ldots, s\}$,

$$b_j = 0 \quad \text{for } j \in T \quad \text{and} \quad a_{ij} = 0 \quad \text{for } i \notin T,\ j \in T.$$

Otherwise it is called *DJ*-irreducible.

Lemma 3.1 (*See Hairer and Wanner [13]*). *A DJ-irreducible, algebraically stable Runge–Kutta method (A, b^{T}, c) satisfies*

$$b_i > 0 \quad \textit{for } i = 1, 2, \ldots, s.$$

It is convenient to introduce the following notation. If D is an $s \times s$ non-negative diagonal matrix, define a pseudo-inner product on $\mathbf{H}^s$ by

$$\langle Y, Z\rangle_D = \sum_{j=1}^{s} d_j \langle Y_j, Z_j\rangle, \quad Y = [Y_1, \ldots, Y_s]^{\mathrm{T}} \in \mathbf{H}^s, \quad Z = [Z_1, \ldots, Z_s]^{\mathrm{T}} \in \mathbf{H}^s$$

and the corresponding pseudo-norm on $\mathbf{H}^s$ by

$$\|Y\|_D = \sqrt{\langle Y, Y\rangle_D}.$$

In particular, when D is the identity matrix I, $\|\cdot\|_D$ is denoted by $\|\cdot\|_I$ for short.

Now we discuss the dissipativity of algebraically stable Runge–Kutta methods for dynamical systems and give the following result.

Theorem 3.2. *Assume that a Runge–Kutta method (A, b^{T}, c) is DJ-irreducible, algebraically stable and satisfies the condition*

$$A^{-1} \textit{ exists and } |1 - b^{\mathrm{T}}A^{-1}e| < 1.$$

Then the method (A, b^{T}, c) is (weakly) $E(\lambda)$-dissipative, where

$$\lambda = \frac{\omega(1 - |1 - b^{\mathrm{T}}A^{-1}e|)^2}{\|b^{\mathrm{T}}A^{-1}\|_2^2}, \quad \omega = \min_{1\leq i\leq s} b_i.$$

Proof. We shall prove the weak dissipativity of the numerical method. The algebraic stability of the method implies that (see Burrage and Butcher [12,14])

$$\|y_{n+1}\|^2 \le \|y_n\|^2 + 2h\sum_{j=1}^{s} b_j \mathcal{R}e\langle Y_j^{(n)}, f(t_n + c_j h, Y_j^{(n)}, y_l, \bar{y}_l)\rangle. \tag{3.4}$$

Then it follows from Conditions 2 that

$$\|y_{n+1}\|^2 \le \|y_n\|^2 + 2\alpha h\|Y^{(n)}\|_B^2 + 2h\beta\sum_{j=1}^{s} b_j\|f(t_n + c_j h, 0, y_l, \bar{y}_l)\|^2, \tag{3.5}$$

where $B = \mathrm{diag}(b_1, \ldots, b_s)$, $Y^{(n)} = [Y_1^{(n)}, \ldots, Y_s^{(n)}]^{\mathrm{T}}$. Therefore, given any $\epsilon > 0$ it follows that either

$$\|y_{n+1}\|^2 \le \|y_n\|^2 + 2\alpha h\epsilon, \tag{3.6}$$

or, by (3.5),

$$\|Y^{(n)}\|_B^2 \le \frac{\beta}{-\alpha}\sum_{j=1}^{s} b_j\|f(t_n + c_j h, 0, y_l, \bar{y}_l)\|^2 + \epsilon. \tag{3.7}$$

Suppose that (3.7) holds. Then from Conditions 2, one easily gets

$$\frac{\beta}{-\alpha}\sum_{j=1}^{s} b_j\|f(t_n + c_j h, 0, y_l, \bar{y}_l)\|^2 \le \frac{\beta}{-\alpha}\sum_{j=1}^{s} b_j(\gamma_2 + L_u\|y_l\|^2 + L_v\|\bar{y}_l\|^2). \tag{3.8}$$

On the other hand, from Conditions 2 and (3.1), we have

$$\|\bar{y}_l\|^2 \le \frac{\gamma_1 + \gamma_2\sigma}{1 - L_v\sigma} + \frac{L_y + L_u\sigma}{1 - L_v\sigma}\|y_l\|^2$$

and hence

$$\begin{aligned}\|Y^{(n)}\|_B^2 &\le \frac{\beta}{-\alpha}\left(\frac{\gamma_2 + L_v\gamma_1}{1 - L_v\sigma} + \frac{L_u + L_v L_y}{1 - L_v\sigma}\|y_l\|^2\right) + \epsilon \\ &\le \tilde{\vartheta}\|y_l\|^2 + \tilde{\upsilon} + \epsilon,\end{aligned} \tag{3.9}$$

where

$$\tilde{\vartheta} = \frac{\beta(L_u + L_v L_y)}{-\alpha(1 - L_v\sigma)}, \quad \tilde{\upsilon} = \frac{\beta(L_v\gamma_1 + \gamma_2)}{-\alpha(1 - L_v\sigma)}.$$

Therefore, it follows from Lemma 3.1 that

$$\|Y^{(n)}\|_I \le \vartheta\|y_l\| + \upsilon + \sqrt{\epsilon/\omega}, \tag{3.10}$$

where

$$\vartheta = \sqrt{\tilde{\vartheta}/\omega}, \quad \upsilon = \sqrt{\tilde{\upsilon}/\omega}.$$

On the other hand, (3.1) implies that

$$y_{n+1} = (1 - b^{\mathrm{T}}A^{-1}e)y_n + (b^{\mathrm{T}}A^{-1} \otimes I)Y^{(n)}, \tag{3.11}$$

where $e = [1, 1, \ldots, 1]^{\mathrm{T}}$. Take the norms of both sides of (3.11) to obtain

$$\begin{aligned}\|y_{n+1}\| &\le |1 - b^{\mathrm{T}}A^{-1}e|\|y_n\| + \|b^{\mathrm{T}}A^{-1}\|_2\|Y^{(n)}\|_I \\ &\le |1 - b^{\mathrm{T}}A^{-1}e|\|y_n\| + \|b^{\mathrm{T}}A^{-1}\|_2\left(\vartheta\|y_l\| + \upsilon + \sqrt{\epsilon/\omega}\right).\end{aligned} \tag{3.12}$$

Let us introduce the notation

$$\delta = |1 - b^{\mathrm{T}}A^{-1}e|, \quad \nu = \|b^{\mathrm{T}}A^{-1}\|_2.$$

Since condition (3.3) implies $\frac{\nu\vartheta}{1-\delta} < 1$, from (3.12) we have

$$\begin{aligned}\|y_{mk}\| &\leq \left(\delta^k + \sum_{i=0}^{k-1}\delta^i \nu\vartheta\right)\|y_{(m-1)k}\| + \sum_{i=0}^{k-1}\delta^i \nu\left(\upsilon + \sqrt{\epsilon/\omega}\right)\\ &\leq \left(\delta^k + \frac{1-\delta^k}{1-\delta}\nu\vartheta\right)\|y_{(m-1)k}\| + \frac{1-\delta^k}{1-\delta}\nu\left(\upsilon + \sqrt{\epsilon/\omega}\right)\\ &\leq \Theta^m\|y_0\| + \frac{1-\Theta^m}{1-\Theta}\frac{1-\delta^k}{1-\delta}\nu\left(\upsilon + \sqrt{\epsilon/\omega}\right),\end{aligned}$$

where

$$\Theta = \delta^k + \frac{\nu\vartheta}{1-\delta}(1-\delta^k) < \delta^k + 1 - \delta^k = 1.$$

Then for any $\varepsilon > 0$ there exists $n_0(\|y_0\|, \varepsilon)$ such that

$$\|y_n\| \leq \frac{\nu\upsilon}{1-\delta-\nu\vartheta} + \varepsilon = R + \varepsilon, \quad n \geq n_0. \tag{3.13}$$

A combination of (3.6) and (3.13) shows that the method (3.1) is weakly $E(\lambda)$-dissipative and hence $B(0, R+\varepsilon)$ is an absorbing set. The proof of the assertion of $E(\lambda)$-dissipativity of the numerical method is the same. □

Corollary 3.3. *Assume that a Runge–Kutta method (A, b^{T}, c) is DJ-irreducible, algebraically stable and satisfies the condition*

$$A^{-1} \text{ exists and } |1 - b^{\mathrm{T}}A^{-1}e| < 1, \quad |1 - b^{\mathrm{T}}A^{-1}e| + \|b^{\mathrm{T}}A^{-1}\|_2 \leq 1.$$

Then the method (A, b^{T}, c) is (weakly) $E(\omega)$-dissipative, where $\omega = \min_{1\leq i\leq s} b_i$.

Proof. The proof is an immediate consequence of the fact that

$$\frac{\|b^{\mathrm{T}}A^{-1}\|_2}{1-|1-b^{\mathrm{T}}A^{-1}e|} \leq 1 \quad \text{and} \quad \frac{\tilde{\vartheta}\|b^{\mathrm{T}}A^{-1}\|_2^2}{\omega(1-|1-b^{\mathrm{T}}A^{-1}e|)^2} \leq \frac{\tilde{\vartheta}}{\omega} < 1. \quad \square$$

We observe that if $a_{si} = b_i,\ i = 1, 2, \ldots, s$, from (3.9), we can obtain the same result. The following theorem states the conclusion.

Theorem 3.4. *Assume that a Runge–Kutta method (A, b^{T}, c) is DJ-irreducible, algebraically stable and satisfies the condition*

$$a_{si} = b_i, \quad i = 1, 2, \ldots, s. \tag{3.14}$$

Then the method (A, b^{T}, c) is (weakly) $E(\lambda)$-dissipative, where $\lambda = b_s$.

Proof. When (3.14) holds, $y_{n+1} = Y_s^{(n)}$. Then from (3.9), we have

$$\|Y_s^{(n)}\|^2 \leq \frac{\tilde{\vartheta}}{b_s}\|y_l\|^2 + \frac{\tilde{\upsilon}}{b_s} + \frac{\epsilon}{b_s}$$

and therefore

$$\|y_{(m-1)k+i}\|^2 \leq \hat{\vartheta}\|y_{(m-1)k}\|^2 + \hat{\upsilon} + \hat{\epsilon}, \quad i = 1, 2, \ldots, k,$$

where

$$\hat{\vartheta} = \frac{\tilde{\vartheta}}{b_s}, \quad \hat{\upsilon} = \frac{\tilde{\upsilon}}{b_s}, \quad \hat{\epsilon} = \frac{\epsilon}{b_s}.$$

Consequently, it follows that

$$\begin{aligned}\|y_{mk}\|^2 &\leq \hat{\vartheta}\|y_{(m-1)k}\|^2 + \hat{\upsilon} + \hat{\epsilon}\\ &\leq \hat{\vartheta}^m\|y_0\|^2 + \frac{1-\hat{\vartheta}^m}{1-\hat{\vartheta}}(\hat{\upsilon} + \hat{\epsilon}).\end{aligned}$$

Note that in this case $\hat{\vartheta} < 1$. As in the proof of Theorem 3.2, from the fact that for any $\varepsilon > 0$ there exists $n_0(\|y_0\|, \varepsilon)$ such that

$$\|y_n\| \le \sqrt{\frac{\tilde{\upsilon}}{b_s - \tilde{\vartheta}}} + \varepsilon, \quad n \ge n_0,$$

we can easily obtain the theorem. $\square$

Example 3.1. The one-leg θ-method

$$y_{n+1} = y_n + hf((1-\theta)t_n + \theta t_{n+1}, (1-\theta)y_n + \theta y_{n+1}), \quad \theta \ge \frac{1}{2}, \tag{3.15}$$

can be written as Runge–Kutta methods with the form (see, e.g., [15])

$$\begin{array}{c|c} \theta & \theta \\ \hline & 1 \end{array}$$

and hence is algebraically stable. It is easy to verify that $\delta = \frac{1-\theta}{\theta}$ and $\nu = \frac{1}{\theta}$. In consequence of Theorem 3.2, the one-leg θ-method (3.15) with $\theta > \frac{1}{2}$ is (weakly) $E((2\theta-1)^2)$-dissipative and $B(0, \frac{\upsilon}{2\theta-1-\vartheta} + \varepsilon)$ is an absorbing set. In particular, the backward Euler method is (weakly) E-dissipative and $B(0, \frac{\upsilon}{1-\vartheta} + \varepsilon)$ is an absorbing set.

Example 3.2. The Radau IA method

$$\begin{array}{c|cc} 0 & 1/4 & -1/4 \\ 2/3 & 1/4 & 5/12 \\ \hline & 1/4 & 3/4 \end{array} \tag{3.16}$$

is algebraically stable. Since $\delta = 0$ and $\nu = 5/2$, from Theorem 3.2, we know that Radau IA method (3.16) is (weakly) $E(1/10)$-dissipative and $B(0, \frac{\upsilon}{1-\vartheta} + \varepsilon)$ is an absorbing set.

Example 3.3. Since all the s $(s \ge 1)$ stage Radau IIA and s $(s \ge 2)$ stage Lobatto IIIC methods satisfy the assumption of Theorem 3.4 with $a_{si} = b_i$, $i = 1, 2, \ldots, s$, we can assert that they are (weakly) $E(\lambda)$-dissipative with $\lambda = b_s$ and $B(0, \sqrt{\tilde{\upsilon}/(b_s - \tilde{\vartheta})} + \varepsilon)$ is an absorbing set.

4. Conclusions

In this short work, we give two sufficient conditions for the dissipativity of neutral differential equations with piecewise constant delay. We also prove that under one of the following two conditions:

(i) A^{-1} exists and $|1 - b^{\mathrm{T}} A^{-1} e| < 1$;
(ii) $a_{si} = b_i$, $i = 1, 2, \ldots, s$,

a *DJ*-irreducible, algebraically stable Runge–Kutta method is (weakly) $E(\lambda)$-dissipative.

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